



RF-3788

M. Sc. (Part-I) Examination

April / May – 2010

Mathematics : 402

(Functional Analysis)

Time : 3 Hours]

[Total Marks : 70

Instructions :

(1)

नीचे दर्शायेव निशा-नीवाणी विगतो उत्तरवडी पर अवश्य लपवी.  
Fillup strictly the details of signs on your answer book.

Name of the Examination :  
M. Sc. (Part-1)

Name of the Subject :  
Mathematics - 402

Subject Code No. : 3 7 8 8 Section No. (1, 2,.....): Nil

Seat No. :

Student's Signature

(2) All questions are compulsory.

(3) Figures to be right of respective question denote the marks of that question.

(4) Standard notations are followed.

1 (a) Prove that  $R^n$  is complete space. 5

(b) Define equivalent norm. If  $\|\cdot\|$  and  $\|\cdot\|_0$  are equivalent norms on X then show that Cauchy sequence in  $(X, \|\cdot\|)$  and  $(X, \|\cdot\|_0)$  are same. 5

(c) Prove that dual space of  $l^1$  is  $l^\infty$ . 4

OR

1 (a) Prove that space  $l^p$  ( $p \geq 1$ ) is complete. 5

(b) Define compact subset. Prove that a compact subset M of a metric space X is closed and bounded. 5

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[Contd...

- (c) Let  $X = V_n(F)$  is an  $n$ -dimensional vector space over the field of scalars  $F$  and  $Y$  be a proper subspace of  $X$  then prove that  $\dim Y < \dim X = n$ . 4
- 2 (a) Show that a finite dimensional vector space is algebraically reflexive. 5
- (b) If  $T$  be a bounded linear operator then prove the following. 5
- (i)  $x_n \rightarrow x \Rightarrow Tx_n \rightarrow Tx$
- (ii) The null space  $N(T)$  is closed.
- (c) State and prove Riesz's lemma. 4

OR

- 2 (a) Define bounded linear operator. If a normed space  $X$  is finite then prove that every linear operator on  $X$  is bounded. 5
- (b) Let  $f: C[a, b] \rightarrow R$  be defined as,  $f(x) = \int_a^b x(t) dt$ . 5
- Show that  $f$  is linear and bounded. Also show that it has a norm  $\|f\| = b - a$ .
- (c) Define integral operator. Prove that integral operator is linear. 4
- 3 (a) Let  $H_1$  and  $H_2$  be two Hilbert space and  $T: H_1 \rightarrow H_2$  be a bounded linear operator. Then prove that Hilbert adjoint operator  $T^*$  of  $T$  exists, and it is unique and bounded linear operator. Also show that  $\|T\| = \|T^*\|$ . 5
- (b) Prove that every bounded linear functional  $f$  on a Hilbert space  $H$  can be represented in terms of inner product as  $f(x) = \langle x, z \rangle$  where  $z$  depends on  $f$  uniquely determined by  $f$  with  $\|z\| = \|f\|$ . 5

- (c) If  $X$  be a complex inner product space for  $x, y \in X$  4  
 then show that,  $\operatorname{Re} \langle x, y \rangle = \frac{1}{4} [\|x+y\|^2 - \|x-y\|^2]$ .

**OR**

- 3 (a) State and prove Bessel's inequality. 5  
 (b) Let  $M$  be a subset of an inner product space  $X$ . 5  
 Then prove that if  $M$  is total in  $X$  then there does not exist a nonzero  $x \in X$  which is orthogonal to every element of  $M$  i.e.  $x \perp M \Rightarrow x = 0$ .  
 (c) If  $X$  be a complex inner product space for  $x, y \in X$ , 4  
 then show that  $\operatorname{Im} \langle x, y \rangle = \frac{1}{4} [\|x+iy\|^2 - \|x-iy\|^2]$ .

- 4 (a) Let  $X$  be Banach space and  $Y$  be a norm space. 10  
 Let  $T_n$  be a sequence of bounded linear operator  
 $T_n : X \xrightarrow{\text{into}} Y$  such that  $\text{seq. } (\|T_n x\|)$  is bounded  $\forall x \in X$ .  
 Then prove that  $\text{seq. } (\|T_n\|)$  is bounded.

- (b) Show that for every  $x$  in a normed space  $X$  has

$$\|x\| = \sup_{\substack{f \in X^1 \\ f \neq 0}} \frac{|f(x)|}{\|f\|}$$

and if  $x_0$  is such that  $f(x_0) = 0$  for all  $f \in X^1$  then show  $x_0 = 0$ .

**OR**

- 4 (a) Let  $X$  and  $Y$  be two Banach space and  $T : D(T) \rightarrow Y$  10  
 be closed linear operator with  $D(T) \subset X$ . If  $D(T)$  is closed in  $X$  then prove that operator  $T$  is bounded.

- (b) Let  $X$  be a normed linear space and  $x_0 \neq 0 \in X$  be arbitrary then prove that  $\exists$  a bounded linear functional  $\hat{\rho}$  on  $X$  such that  $\|\hat{\rho}\| = 1$  and  $\hat{\rho}(x_0) = \|x_0\|$ . 4
- 5 (a) State and prove Banach fixed point theorem. 5
- (b) Write a note on Gauss seidel iteration method. 5
- (c) Let  $X$  and  $Y$  be inner product space and  $Q: X \rightarrow Y$  a bounded linear operator then prove the following 4
- (i)  $Q=0$  if and only if  $\langle Qx, Y \rangle = 0$  for all  $x \in X + y \in Y$ .
- (ii) If  $Q: X \rightarrow Y$  when  $X$  is complex and  $\langle Qx, x \rangle = 0$  for all  $x \in X$  then show that  $Q=0$ .

**OR**

- 5 (a) Write a note on Jacobi iteration method. 5
- (b) State and prove fixed point lemma. 5
- (c) Let  $X = \{x \in R | x \geq 0\} \subset R$  and let the mapping 4
- $T: X \rightarrow X$  be defined by  $T_x = \frac{x}{2} + x^{-1}$ . Show that  $T$  is a contraction and also find the smaller  $\alpha$ .